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Stringy profiles of gauge field tadpoles near orbifold singularities
1. Heterotic string calculations

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Abstract: Closed string theories on orbifolds contain both untwisted and twisted states. The latter are normally assumed to live exactly at the orbifold fixed points. We perform a calculation of a gauge field tadpole amplitude and show that off-shell both the twisted and untwisted states give rise to non-trivial momentum profiles over the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. These profiles take the form of gaussian distributions integrated over the fundamental domain of the modular parameter of the torus. The propagators of the internal coordinate fields on the torus world sheet determine the width of the gaussian profiles. These propagators are determined up to a single normal ordering constant which must be bounded below to allow the existence of the coordinate space representation of these profiles. Apart from the expected massless states, massive and even tachyonic string excitations contribute to the profiles in some anomalous U(1) models. However, when the tadpole is integrated over the internal dimensions, these tachyonic contributions cancel in a non-trivial manner.

Keywords: Field Theories in Higher Dimensions, Superstrings and Heterotic Strings, Superstring Vacua, String Field Theory.

Contents

1. Introduction

In this series of two papers we investigate some local properties of string theory on orbifold singularities and their interpretation in field theory. Due to pioneering investigations [1, 2, 3] of strings on orbifolds, it is commonly accepted that strings can propagate without difficulty on a target space containing orbifold singularities. This has been confirmed by many investigations of zero mode properties of various types of string theories on different orbifold backgrounds. Field theories on orbifolds always seem to require more care, particularly when gravitational effects are involved. Close to the orbifold singularity, strong curvature effects can become important and can lead to a breakdown of effective field theory methods. We therefore expect that studying strings in the background of an orbifold singularity might teach us something about how we should treat those singularities in field theory.

For both field and string theoretical reasons we employ gauge field tadpoles as local probes of orbifold singularities in these two papers. Field theory arguments suggested that a one-loop Fayet-Iliopoulos D-term tadpole [4, 5] would be generated in heterotic string models with an anomalous U(1) [6]. This was soon confirmed by direct string calculations [7, 8]. (A similar investigation has been performed for orbifolded type-I string theories [9].) It was recently realized that in five dimensional field theory models on $S^1/\mathbb{Z}_2$ [10]–[14], higher dimensional field theory models [15, 16], as well as in heterotic string inspired models [17] that these tadpoles are generated locally at orbifold fixed points. By supersymmetry these D-term tadpoles are accompanied by tadpoles for the internal U(1) gauge field strength at

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those singularities. (In the case of five dimensional models this reduces to a tadpole for the derivative of the real scalar contained in the vector multiplet.) In this paper we propose to verify the existence of these gauge field tadpoles in string theory, and to investigate their properties.

In field theory it has been shown that these gauge tadpoles can lead to (strong) localization effects over the extra dimensions [13, 14, 18, 19], which can be important for phenomenological applications of higher dimensional models. The physical interpretation of these field theory models is rather involved, since these tadpoles are proportional to (derivatives of) delta functions at the fixed points. The energies needed to resolve such singularities are much higher than the assumed validity of the effective field theory. However, as string theory is assumed to be a complete theory, it should solve this problem by introducing some sort of ultra-violet cut-off. In addition, the tadpole contributions due to bulk and fixed point fields seem quite different in (string-inspired) field theory: The former have profiles depending on the internal momenta, and therefore extend into the extra dimensions, while the latter are assumed to be confined exactly at the orbifold singularities. As the corresponding untwisted and twisted states in string theory are related to each other by modular transformations, it is interesting to see to what extent these differences persist.

We have decided to divide the presentation of our results into two separate publications. This paper focuses on the details of the string calculation that establishes that the gauge field tadpole discussed in [17] also arises in string theory. As the physical interpretations of these results are of interest to a wider audience of both string and field theorists, we postpone our detailed comparison between the string and field theory results to the accompanying paper [20].

We have restricted ourselves to the well-studied heterotic $\mathrm{E}_8 \times \mathrm{E}_8{}'$ string on the non-compact six dimensional orbifold $\mathbb{C}^3/\mathbb{Z}_3$. This choice is motivated by a number of factors: It is well-know that string models with an anomalous U(1) in their zero mode spectrum exist on $\mathbb{Z}_3$ orbifolds [21]. As discussed above, on field theoretical grounds we would expect that tadpoles for internal gauge fields are also generated within the string setting. By considering this simple non-compact orbifold we avoid the additional complications of winding modes and Wilson lines. A compact orbifold would introduce at least one more scale (its volume) and this would make it more difficult to trace how string theory introduces a regularization of the delta-like singularities. We are also motivated to use the heterotic string in light of some recent interest in its compactifications. The authors of [22] find that it is possible to stabilize moduli of compactifications on Calabi-Yau threefolds. Many six dimensional orbifolds are singular limits of Calabi-Yau spaces, and so these spaces can be thought of as providing a regularization of the orbifold singularity.

The tadpole profile over the orbifold will suggest that the parallels between string and field theory can even be extended to off-shell amplitudes. Strictly speaking string amplitudes are only defined on-shell, but as off-shell amplitudes in field theory contain a wealth of extra information, many authors have pursued the development of off-shell string theory [23]–[29]. This progress was partly stimulated by applying stringy techniques in field theory calculations [30, 31]. Moreover, to understand the dynamics of tachyons in unstable brane configurations, an off-shell description is essential [32, 33]. (Tachyons may

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also arise in closed string theory [34].) Off-shell amplitudes become dependent on normal ordering constants. These constants can be interpreted as coefficients of conformal maps of the worldsheet torus to itself [30, 35]. From the very outset our investigation had a different objective than those works: They aim to describe the string (zero) modes off-shell, while we are interested in the momentum dependence in the extra dimensions. For a tadpole on an $\mathbb{Z}_3$ orbifold, no four dimensional momenta can flow in or out, therefore probing this dependence necessarily requires going off-shell.

This paper is organized as follows: In section 2 we define our notation for the heterotic string on the non-compact orbifold $\mathbb{C}^3/\mathbb{Z}_3$. In section 3 we give the string calculation of the gauge tadpole at one-loop in string perturbation theory. This calculation identifies the twisted propagators as the characteristic widths of the (off-shell) momentum profiles for the various states on the orbifold. The twisted propagators are determined up to a single normal ordering constant in section 4. The properties of the twisted propagators lead to a natural classification of inequivalent sectors which give different contribution to the profiles of the local gauge field tadpoles. In section 5 we investigate the various orbifold models to determine what type of states (tachyonic, massless or massive) contribute to these tadpoles within those sectors. Our conclusions are summarized in section 6. In appendix A several useful properties of fermionic and bosonic world sheet theories are reviewed, and appendix B gives a brief overview of our conventions for the elliptic functions that we use throughout the paper.

2. Heterotic string on $\mathbb{C}^3/\mathbb{Z}_3$

In section 3 we will perform a one-loop calculation in heterotic string theory of gauge field tadpoles on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$; here we present the framework for that computation. We consider the world sheet torus parameterized by the complex coordinate σ with Teichmüller parameter τ; the torus periodicities are defined as $\sigma \sim \sigma + 1$ and $\sigma \sim \sigma + \tau$. On this world sheet torus the conformal field theories live in the heterotic string. We describe them in light-cone gauge: $X^M(\sigma), M = 2, \ldots 9$ are the coordinate fields and $\psi^M(\sigma)$ their right-moving fermionic partners. The part of the theory encoding the gauge structure is described by the left-moving fermions $\lambda_a^{2I}(\sigma), \lambda_a^{2I+1}(\sigma)$. For the $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory there are two sets, labeled by $a = 1, 2$, of $I = 1, \ldots 8$ fermions. (The SO(32) string contains $I = 1, \ldots 16$ fermions and $a = 1$. As most of our description applies to both theories, we can use both indices I and a to describe the SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory simultaneously.) For strings on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ it is convenient to define complex combinations of these fields

$$X_i = \frac{1}{\sqrt{2}}(X_{2i+2} + iX_{2i+3})\,, \qquad \psi_i = \frac{1}{\sqrt{2}}(\psi_{2i+2} + i\psi_{2i+3})\,, \qquad \lambda_a^I = \frac{1}{\sqrt{2}}(\lambda_a^{2I} + i\lambda_a^{2I+1}) \tag{2.1}$$

and their complex conjugates for $i = 0, \ldots 3$ in light-cone gauge. The boundary conditions of these fields on the world sheet torus are given by

$$\begin{aligned}
X_i(\sigma+1) &= e^{-2\pi i\, p\phi_i} X_i(\sigma)\,, & X_i(\sigma+\tau) &= e^{+2\pi i\, p'\phi_i} X_i(\sigma),\\
\psi_i(\sigma+1) &= e^{-2\pi i(p\phi_i + s/2)}\psi_i(\sigma), & \psi_i(\sigma+\tau) &= e^{+2\pi i(p'\phi_i + s'/2)}\psi_i(\sigma)\,,\\
\lambda_a^I(\sigma+1) &= e^{-2\pi i(pv_a^I + t_a/2)}\lambda_a^I(\sigma)\,, & \lambda_a^I(\sigma+\tau) &= e^{+2\pi i(p'v_a^I + t_a'/2)}\lambda_a^I(\sigma)\,,
\end{aligned} \tag{2.2}$$

where $p, p' = 0, 1, 2$ label the different orbifold boundary conditions, and $s, s', t_a, t'_a = 0, 1$ define the different spin structures for ψ_i and λ^I_a, respectively. The boundary conditions (2.2) are fully specified by giving the spacetime and gauge shifts ϕ_i and v^I_a, respectively. For the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ we have the requirements [1, 2]

$$3\phi_i = 0 \mod 1\,, \quad \frac{3}{2}\sum_i \phi_i = 0 \mod 1\,, \quad 3v^I_a = 0 \mod 1\,, \quad \frac{3}{2}\sum_{a,I} v^I_a = 0 \mod 1\,. \tag{2.3}$$

Since we do not orbifold the four dimensional spacetime, we take $\phi_0 = 0$. By integral shifts and Weyl reflections, we can always bring ϕ to the form $\phi = (0, 1, 1, \text{-}2)/3$, so that $\sum_i \phi_i = 0$. Similarly, we will assume that gauge shifts have been taken such that $\sum_{a,I} v^I_a = 0$. These constraints on the spacetime and gauge shifts complete the definition of the different world sheet theories of the orbifold model.

Each different boundary condition, encoded by p, p', s, s', t_a, t'_a, defines a different world sheet theory. The full string theory is obtained by combining all these possible boundary conditions. The choices of relative phases between the sectors define GSO projections [36], for the orbifold and three different spin structures [37]. One defines the full string partition function as the sum

$$Z = \sum_{p,p',t_a,t'_a} Z^{p,p'}_{t_a,t'_a}\,, \qquad Z^{p,p'}_{t_a,t'_a} = \sum_{s,s'} \tilde{\eta}^{p,p',\,s,s'}_{t_b,t'_b}\; Z^{p,p',s,s'}_{t_a,t'_a}\,, \tag{2.4}$$

$$Z^{p,p',s,s'}_{t_a,t'_a} = \mu_0\, e^{-2\pi i \sum_{a,I} v^I_a\, p \frac{t'_a}{2}} \prod_{i=0}^{3} Z_X \begin{bmatrix} p\,\phi_i \\ p'\phi_i \end{bmatrix} (\mu_i|\tau)\, Z_\psi \begin{bmatrix} p\,\phi_i + \frac{s}{2} \\ p'\phi_i + \frac{s'}{2} \end{bmatrix} (\mu_i|\tau) \prod_{a,I} Z_\lambda \begin{bmatrix} p\,v^I_a + \frac{t_a}{2} \\ p' v^I_a + \frac{t'_a}{2} \end{bmatrix} (\nu^I_a|\tau)$$

over the partition functions of the individual sectors. These partition functions are given by

$$Z_X \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\mu|\tau) = \frac{\big|\eta(\tau)\big|^2}{\Big|\vartheta \begin{bmatrix} \frac{1}{2}-\alpha \\ \frac{1}{2}-\beta \end{bmatrix} (\mu|\tau)\Big|^2}\,, \qquad Z_\psi \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\bar{\mu}|\bar{\tau}) = \frac{\overline{\vartheta \begin{bmatrix} \frac{1}{2}-\alpha \\ \frac{1}{2}-\beta \end{bmatrix} (\mu|\tau)}}{\overline{\eta(\tau)}}\,,$$

$$Z_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\nu|\tau) = \frac{\vartheta \begin{bmatrix} \frac{1}{2}-\alpha \\ \frac{1}{2}-\beta \end{bmatrix} (\nu|\tau)}{\eta(\tau)}\,, \tag{2.5}$$

where we have introduced the complex sources μ_i, ν^I_a for later convenience. We have collected many useful details of these twisted world sheet theories in appendix A. Defining the partition function as the limit of $\mu, \nu \to 0$ avoids having to treat the case $\alpha = \beta = 0$ separately, provided that the total partition function is multiplied by μ_0. These partition functions have been computed in (A.14) and (A.7). (Since the fields ψ_i are right-movers we use the complex conjugate of the left-moving result.) The phase factor in (2.4) have been introduced to ensure that the individual partition functions $Z^{p,p',s,s'}_{t_a,t'_a}$ are functions of the equivalence classes of $p \sim p+3$ and $s \sim s+2$, $t_a \sim t_a + 2$ [38, 39, 40].

We impose the standard GSO projections for the right-moving fermions ψ and the left-moving fermions λ_a. In addition, we enforce a generalized projection required by the orbifold boundary conditions. The compatibility of these projections is encoded in the factorization of the phases as

$$\begin{aligned}\eta^{p,p',\,s,s'}_{t_a,t'_a} &= \eta^{s,s'}\,\tilde{\eta}^{p,p'}_{t_a,t'_a}\,, \qquad \eta^{s,s'} = \exp\{-\pi i\, ss'\},\\ \eta^{p,p'}_{t_a,t'_a} &= \exp 2\pi i \left\{ \frac{1}{2}\left(\sum_i (\phi_i)^2 - \sum_{a,I}(v^I_a)^2\right) pp' \right\}.\end{aligned} \tag{2.6}$$

These phases are consequences of modular invariance, as been investigated by various groups [41, 42, 43, 39, 38, 44, 40]. In our case we find that invariance under $\tau \to \tau+1$ and $\tau \to -1/\tau$ leads to the relations between the phases

$$\begin{aligned}\tilde{\eta}^{p,p'+p,\,s,s'+s}_{t_a,t'_a+t_a} &= e^{2\pi i\left\{\sum\limits_{a,I} v^I_a p \frac{t_a}{2}\right\}}\; e^{2\pi i\left\{\frac{1}{2}\sum\limits_i \left(p\phi_i+\frac{s}{2}\right)^2 - \frac{1}{2}\sum\limits_{a,I}\left(pv^I_a+\frac{t_a}{2}\right)^2\right\}}\;\eta^{p,p',\,s,s'}_{t_a,t'_a}\,,\\ \tilde{\eta}^{-p',p,-s',s}_{-t'_a,t_a} &= e^{2\pi i\left\{-\sum\limits_{a,I} v^I_a\left(p\frac{t'_a}{2}+p'\frac{t_a}{2}\right)\right\}}\times\\ &\quad\times e^{2\pi i\left\{\sum\limits_i\left(\frac{1}{2}+p\phi_i+\frac{s}{2}\right)\left(\frac{1}{2}-p'\phi_i-\frac{s'}{2}\right)-\sum\limits_{a,I}\left(\frac{1}{2}+pv^I_a+\frac{t_a}{2}\right)\left(\frac{1}{2}-p'v^I_a-\frac{t'_a}{2}\right)\right\}}\;\eta^{p,p',\,s,s'}_{t_a,t'_a}\,,\end{aligned} \tag{2.7}$$

respectively. The first terms in these exponents arise because of the phase in the partition functions (2.4). Consistency of the solution (2.6) is ensured by requiring that the following conditions on the spacetime and gauge shifts are fulfilled

$$\frac{1}{2}\sum_{a,I} v^I_a - \frac{1}{2}\sum_i \phi_i = 0 \mod 1\,, \qquad \frac{3}{2}\sum_{a,I}(v^I_a)^2 - \frac{3}{2}\sum_i(\phi_i)^2 = 0 \mod 1\,. \tag{2.8}$$

The second condition follows upon requiring modular invariance under the transformation $\tau \to \tau+3$ in the sector $s = t_a = 0$, see [45, 46].

The sum over the spin structures s, s' can be removed by applying the Riemann's identity (B.8), and we find

$$Z^{p,p'}_{t_a,t'_a}(\mu,\nu|\tau) = \tilde{\eta}^{p,p'}_{t_a,t'_a}\frac{2\mu_0}{\eta(\tau)^{12}}\prod_{a,I}\vartheta\begin{bmatrix}\frac{1}{2}-p\,v^I_a-\frac{t_a}{2}\\ \frac{1}{2}-p'v^I_a-\frac{t'_a}{2}\end{bmatrix}(\nu^I_a|\tau)\prod_{i=0}^{3}\frac{\overline{\vartheta\begin{bmatrix}\frac{1}{2}-p\,\phi_i\\ \frac{1}{2}-p'\phi_i\end{bmatrix}(\text{-}\tilde{\mu}_i|\tau)}}{\left|\vartheta\begin{bmatrix}\frac{1}{2}-p\,\phi_i\\ \frac{1}{2}-p'\phi_i\end{bmatrix}(\mu_i|\tau)\right|^2}\,, \tag{2.9}$$

with the definition

$$\tilde{\mu}_i = -\mu_i + \frac{1}{2}\sum_j \mu_j\,. \tag{2.10}$$

Notice that in the limit of $\mu_i \to 0$ the partition function becomes a holomorphic function of τ. (In fact — as is well-known — the whole partition function is zero in this limit, and this is the reason that we have introduced the μ_0.) In this limit the expression for the partition function simplifies to

$$Z^{p,p'}_{t_a,t'_a}(\nu|\tau) = \frac{-1}{2\pi}\tilde{\eta}^{p,p'}_{t_a,t'_a}\frac{1}{\eta(\tau)^{15}}\prod_{a,I}\vartheta\begin{bmatrix}\frac{1}{2}-p\,v^I_a-\frac{t_a}{2}\\ \frac{1}{2}-p'v^I_a-\frac{t'_a}{2}\end{bmatrix}(\nu|\tau)\left(\prod_{i=1}^{3}\vartheta\begin{bmatrix}\frac{1}{2}-p\,\phi_i\\ \frac{1}{2}-p'\phi_i\end{bmatrix}(0|\tau)\right)^{-1}. \tag{2.11}$$

3. String calculation of tadpoles of internal gauge fields

We investigate the local tadpole structure of Cartan gauge fields with internal spacetime indices by calculating the expectation values of the normal ordered vertex operators

$$V_j^{bJ} = :(\bar\partial X_j + i\, k_M \psi^M \psi_j)\, \bar\lambda_b^J \lambda_b^J\, e^{ik_M X^M} \; : \tag{3.1}$$

at the one loop (torus) order. The vertex operators $V_{\underline{j}}^{bJ}$ are defined as V_j^{bJ} with the replacement $j \to \underline{j}$. As the expectation values of these vertex operators are closely related, we focus on one of them only, and make their relation explicit at the point where this is most convenient. No sum over b or J is implied in (3.1), and the combination of gauge indexed fermions $\bar\lambda_b^J \lambda_b^J$ has been chosen so that the vertex operator is in the Cartan subalgebra of the gauge group. We calculate the expectation value of the vertex operator as a weighted average with respect to the partition functions (2.4). We find that

$$\langle V_j^{bJ}(k)\rangle = ik_{\underline{j}}\, \frac{\delta^4(k_4)}{(2\pi)^4}\, \langle G_j^{bJ}(k_6)\rangle \qquad \langle G_j^{bJ}(k_6)\rangle = \sum Z^{p,p',s,s'}_{t_a,t'_a}\, \langle G_j^{bJ}(k_6)\rangle^{p,p',s,s'}_{t_a,t'_a}\,, \tag{3.2}$$

where the sub- and superscripts on the expectation values denote that they are evaluated within the corresponding set of boundary conditions. The subscripts on k_4 and k_6 indicate that these momentum vectors lie in the four dimensional Minkowski or the six dimensional internal space, respectively, and $k = (k_4, k_6)$. The brackets $\langle\rangle$ without subscripts refer to the sum of expectation values in the different sectors weighted by the corresponding partition functions. The four dimensional delta function results from the zero mode integral of $\exp(ik_\mu X^\mu)$ and imposes four dimensional momentum conservation.

The dependence on the bosonic fields X_i is defined by a point splitting regularization in the following manner: We consider exponentials $\bar\partial X_j\, \exp(ik_j X_{\underline{j}} + ik_{\underline{j}} X_j)$, $\bar\partial X_{\underline{j}}\, \exp(ik_j X_{\underline{j}} + ik_{\underline{j}} X_j)$ and $\exp(ik_j X_{\underline{j}} + ik_{\underline{j}} X_j)$ with the bosonic fields $X_j(\sigma)$ and $X_{\underline{j}}(\sigma')$ evaluated at different points $\sigma \neq \sigma'$ on the torus world sheet. Next we use (A.16) to express the expectation values of the exponentials in terms of bosonic propagators defined in (A.15), and finally, we take the limit of zero separation. (One of the two derivatives comes with an opposite sign, as it is a derivative with respect to the second argument of $\tilde\Delta_X$.) In this way we find that in a particular sector

$$\langle G_j^{bJ}\rangle^{p,p',s,s'}_{t_a,t'_a} = \left(-\bar\partial \Delta_X \begin{bmatrix} p\ \phi_j \\ p'\phi_j \end{bmatrix} + \Delta_\psi \begin{bmatrix} p\ \phi_j + \frac{s}{2} \\ p'\phi_j + \frac{s'}{2} \end{bmatrix} \right) \Delta_\lambda \begin{bmatrix} p\ v_b^J + \frac{t_b}{2} \\ p' v_J^b + \frac{t'_b}{2} \end{bmatrix} \prod_{i=1}^{3} e^{-k_{\underline{i}} k_i\, \Delta_X \left[\begin{smallmatrix} p\ \phi_i \\ p'\phi_i \end{smallmatrix}\right]}\,. \tag{3.3}$$

This is expressed in terms of the normal ordered propagators Δ_X, Δ_λ and Δ_ψ at zero world-sheet separation. The fermionic propagators Δ_λ and Δ_ψ are defined by a similar point splitting regularization, and given by (A.10) and its complex conjugate.

We use conformal normal ordering to remove any singularities that may arise in this limit, and this introduces normal ordering constants for the correlator of the bosonic fields. On-shell amplitudes do not depend on these arbitrary constants. Also in the present case we see, that for the on-shell tadpole ($k_i = 0$) the possible normal ordering constants drop out. However, as we are interested in the off-shell properties of the tadpole, we expect to find some dependence on these constants. We return to this point at the end of section 4.

Taking the limit of zero separation on the equation

$$\bar{\partial}\tilde{\Delta}_X \begin{bmatrix} p\ \phi_i \\ p'\phi_i \end{bmatrix} = \tilde{\Delta}_\psi \begin{bmatrix} p\ \phi_i \\ p'\phi_i \end{bmatrix}, \tag{3.4}$$

which can be deduced from (A.8) and (A.15), we re-express (3.3) as

$$\langle G_j^{bJ} \rangle_{t_a,t'_a}^{p,p',s,s'} = \left(-\Delta_\psi \begin{bmatrix} p\ \phi_j \\ p'\phi_j \end{bmatrix} + \Delta_\psi \begin{bmatrix} p\ \phi_j + \frac{s}{2} \\ p'\phi_j + \frac{s'}{2} \end{bmatrix} \right) \Delta_\lambda \begin{bmatrix} p\ v_b^J + \frac{t_b}{2} \\ p' v_J^b + \frac{t'_b}{2} \end{bmatrix} \prod_{i=1}^{3} e^{-k_{\underline{i}} k_i\, \Delta_X \left[\begin{smallmatrix} p\ \phi_i \\ p'\phi_i \end{smallmatrix}\right]}. \tag{3.5}$$

Inserting the expressions for the normal ordered fermionic correlators (A.10) for Δ_λ and the complex conjugate for Δ_ψ and combining this with the character valued partition function (2.9), we find that

$$\langle G_j^{bJ} \rangle = \sum_{p,p',t_a,t'_a} \frac{\partial}{\partial \nu_b^J} \frac{\partial}{\partial \bar{\mu}_j} Z_{t_a,t'_a}^{p,p'}(\mu,\nu) \prod_{i=1}^{3} e^{-\bar{k}_i k_i\, \Delta_X \left[\begin{smallmatrix} p\ \phi_i \\ p'\phi_i \end{smallmatrix}\right]} \Bigg|_{\mu=\nu=0}. \tag{3.6}$$

To write this we have used that the differentiation with respect to $\bar{\mu}_j$ of the partition function $Z_{t_b,t'_b}^{p,p'}$ precisely gives the Δ_{ψ_j} correlators (with and without the spin structures) of (3.5). To evaluate the derivative with respect to $\bar{\mu}_j$ in the limit of $\mu \to 0$, we only need to consider the anti-holomorphic part of (2.9), hence we obtain

$$\bar{\mu}_0 \frac{\partial}{\partial \bar{\mu}_j} Z_{t_a,t'_a}^{p,p'}(\mu,\nu|\tau,\bar{\tau}) \Bigg|_{\mu=0} = \frac{1}{2} Z_{t_a,t'_a}^{p,p'}(\nu|\tau), \tag{3.7}$$

where the holomorphic partition function $Z_{t_a,t'_a}^{p,p'}(\nu|\tau)$ is given in (2.11).The rescaling with $\bar{\mu}_0$ is required, otherwise the result diverges in the limit $\mu \to 0$. The local gauge field tadpole then becomes a function of the holomorphic partition function only, and hence only depends on the $N = 1$ multiplets rather than their individual bosonic and fermionic constituents. This is a general feature of the application of the Riemann's identity (B.8) within string theory. Another way to express this is to say that the sum over the spin structures s gives the sum over space-time bosons and fermions, and thus we have a contribution from both bosons and fermions in what may be thought of as a field theory one-loop diagram.

The expectation values (3.2) of the vertex operators V_j^{bJ} and $V_{\underline{j}}^{bJ}$ can be written as

$$\langle V_j^{bJ}(k) \rangle = i k_{\underline{j}} \frac{\delta^4(k_4)}{(2\pi)^4} \langle G^{bJ}(k_6) \rangle, \qquad \langle V_{\underline{j}}^{bJ}(k) \rangle = -i k_j \frac{\delta^4(k_4)}{(2\pi)^4} \langle G^{bJ}(k_6) \rangle. \tag{3.8}$$

Using (3.7), the functions G^{bJ} do not depend on the internal spacetime indices $j, \underline{j}$. Moreover, the sign of the expectation values of V_j^{bJ} and $V_{\underline{j}}^{bJ}$ are opposite for two reasons: Since ψ_j and $\psi_{\underline{j}}$ are fermions, interchanging their order in (3.1) gives a relative minus sign. Secondly, taking derivatives with respect l and $\bar{l}$ of equation (A.16) gives the expectation values of $\bar{\partial} X_{\underline{j}} \exp(i k_i X_{\underline{i}} + i k_{\underline{i}} X_i)$ and $\bar{\partial} X_j \exp(i k_i X_{\underline{i}} + i k_{\underline{i}} X_i)$ with a relative sign again.

We can Fourier transform $G^{bJ}(k_6)$ over the orbifolded dimensions to obtain $\mathcal{G}^{bJ}(z)$ and integrate the result over the fundamental domain $\mathcal{F}$ of the modular parameter τ, defining

$$\mathcal{G}^{bJ}(z) = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \mathcal{G}^{bJ}(z|\tau). \tag{3.9}$$

These considerations imply that the corresponding effective field theory interaction is given by

$$S_{FI} = \int \mathrm{d}^{10}x \sum_{j,b,J} \big(\partial_{\underline{j}} A_j^{bJ} - \partial_{\underline{j}} A_j^{bJ}\big) \mathcal{G}^{bJ}(z)\,. \tag{3.10}$$

Since only the exponential factor in (3.6) depends on the external six dimensional momentum k, the Fourier transform is obtained easily

$$\mathcal{G}^{bJ}(z|\tau) = \sum_{p,p',t_a,t'_a} \frac{1}{2}\frac{\partial}{\partial \nu_b^J} Z^{p,p'}_{t_a,t'_a}(\nu)\bigg|_0 \prod_{i=1}^{3} \frac{2\pi}{\Delta_X \begin{bmatrix} p\ \phi_i \\ p'\phi_i \end{bmatrix}} e^{-\bar{z}z/\Delta_X \begin{bmatrix} p\ \phi_i \\ p'\phi_i \end{bmatrix}}\,. \tag{3.11}$$

This shows that $\Delta_X(\tau)$ can be interpreted as the width of a complex three dimensional gaussian distribution for a given value of the modular parameter τ. (This is consistent with the observations in [47, 48, 49] that wavefunctions of the gravitational wave states are gaussians of widths specified by the orbifold twist.) The existence of the Fourier transform requires that $\Delta_X(\tau) > 0$ for all τ in the fundamental domain $\mathcal{F}$. According to equation (3.11) the contributions to the profile of the tadpole depend on the boundary conditions corresponding to the distinct sectors of the orbifolding. This gives a measure to what extent the states of these different sectors are localized near the fixed point of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. As this information is encoded in the functions $\Delta_X \begin{bmatrix} p\ \phi_i \\ p'\phi_i \end{bmatrix}$, their computation in section 4 is of central importance.

The integrated tadpole. While the central theme of this work is the local structure of tadpoles, we would like to make a couple of comments about the global properties. The first all, at the zero mode level the gauge field tadpole vanishes trivially, since for a constant gauge field background the field strength (3.10) is identically zero. However, the function multiplying the internal field strength $F^{bJ}_{j\underline{j}} = \partial_j A^{bJ}_{\underline{j}} - \partial_{\underline{j}} A^{bJ}_j$ in that expression can be integrated over the full orbifold

$$\int_{\mathbb{C}^3/\mathbb{Z}_3} \mathrm{d}^6 z\, \mathcal{G}^{bJ}_j(z) = \frac{1}{3}\int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, \frac{1}{2}\frac{\partial}{\partial \nu_b^J} Z(\nu|\tau)\bigg|_0\,. \tag{3.12}$$

This result follows immediately, since (3.11) contains properly normalized gaussian distributions. In addition, the arguments presented in ref. [17] lead us to expect that this integrated tadpole is proportional to the zero mode D-term. Using the method of computing the integral over the fundamental domain of a holomorphic function of τ explained in ref. [50], it follows that only the massless string modes contribute.[1] This means that the integrated tadpole is proportional to the sum of U(1) charges of these zero modes. We will use this as a cross check of our results for the local tadpoles in section 5. In this sense our calculations are a direct extension of the results of Atick et al. [7].

[1]Unfortunately, as the local tadpole (3.11) is not holomorphic in τ because it depends on $\Delta_X(\tau,\bar{\tau})$, such powerful complex function techniques cannot be applied to our local results.

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4. The twisted propagator

In the previous section we found that the twisted propagators can be interpreted as the width of gaussians in momentum or coordinate space characterizing the profiles of the gauge field tadpoles. Therefore, it is important to determine them explicitly for the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. Since for this $\mathbb{Z}_3$ orbifold we can choose $\phi = (\phi_i) = (1, 1, -2)/3$, the three functions $\Delta_X \begin{bmatrix} p\,\phi_i \\ p'\phi_i \end{bmatrix}$ for $i = 1, 2, 3$ are the same; this reflects the rotational symmetry of this orbifold. (This paper specifically focus on the $\mathbb{Z}_3$ orbifold, however, the method of determining the relevant twisted propagators can easily be extended to $\mathbb{Z}_N$ orbifolds.)

The correlator of a boson with non-trivial boundary conditions (A.11), with $\alpha = p/3$, $\beta = p'/3$ and $p, p' \in \{0, 1, 2\}$ and not both, zero reads

$$\tilde{\Delta}_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix} (\sigma|\tau) = -\frac{1}{2\pi} \sum_{m,n} \frac{2\tau_2}{|\tau(m + p/3) + n + p'/3|^2} \Phi \begin{bmatrix} m{+}p/3 \\ n\;{+}p'/3 \end{bmatrix} (-\sigma|\rho)\,. \tag{4.1}$$

Using he mode functions $\Phi \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ given in (A.1) (A.11) we can write the formal series expansion for this correlator as

$$\tilde{\Delta}_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix} (\sigma|\tau) = -\frac{3^2}{2\pi} \sum_{m,n} \frac{2\tau_2 \exp\Big\{ -2\pi i\, \frac{\sigma'(\bar{\tau}(3m+p)+3n+p') - \bar{\sigma}'(\tau(3m+p)+3n+p')}{\bar{\tau}-\tau} \Big\}}{|\tau(3m + p) + 3n + p'|^2}, \tag{4.2}$$

where we have reparameterized $\sigma = 3\sigma'$. We define the projector

$$\delta_3(m) = \frac{1}{3} \sum_{k=0}^{2} e^{2\pi i\, km/3} = \begin{cases} 1, & m = 0 \mod 3\,, \\ 0, & m \neq 0 \mod 3\,, \end{cases} \tag{4.3}$$

and obtain

$$\tilde{\Delta}_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix} (\sigma|\tau) = -\frac{3^2}{2\pi} \sum_{m',n'}{}' \delta_3(m'{-}p)\delta_3(n'{-}p') \frac{2\tau_2 \exp\Big\{ \text{-}2\pi i\, \frac{\sigma'(\bar{\tau}m'+n') - \bar{\sigma}'(\tau m'+n')}{\bar{\tau}-\tau} \Big\}}{|\tau m' + n'|^2}\,. \tag{4.4}$$

The restriction of the sum without $(m', n') = (0, 0)$, denoted by the prime on the sum, is consistent by virtue of the assumption that p and p' are not both zero modulo three. By inserting the definition (4.3) of the projectors the sums over m', n' can be cast in the form of the untwisted correlator (A.17). We find the twisted propagators can be written as sums over untwisted propagators

$$\tilde{\Delta}_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix} (\sigma|\tau) = \sum_{k,l=0}^{2} e^{-2\pi i(pk+p'l)/3}\, \tilde{\Delta} \left(\frac{\sigma + k - l\tau}{3} \Big| \tau \right). \tag{4.5}$$

The correlators in the zero separation limit with all singular terms removed are denoted by Δ without the tilde. For all $(k, l) \neq 0$ the limit $\sigma \to 0$ does not lead to any singularity and can be taken readily. This leaves the case $k = l = 0$, but this one is determined in (A.20). Therefore the expectation value the twisted propagator at zero separation is given by

$$\Delta_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix} (\tau) = \sum_{(k,l)\neq 0} e^{-2\pi i(pk+p'l)/3}\, \tilde{\Delta} \Big(\frac{k - \tau l}{3} \Big| \tau \Big) - \ln(2\tau_2) + \tilde{c}\,. \tag{4.6}$$

The constant $\tilde{c}$ denotes the normal ordering constant for the untwisted propagator. For all normal ordered twisted propagators $\Delta \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix}$ the dependence on the constant $\tilde{c}$ is the same. Not all propagators for $p, p' = 0, 1, 2$ are independent: Using the projector (4.3) and the definition (4.1) in the $\sigma \to 0$ limit it follows immediately that

$$\Delta_X \begin{bmatrix} p/3+1 \\ p'/3 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} p/3 \\ p'/3+1 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} -p/3 \\ -p'/3 \end{bmatrix}(\tau) = \left(\Delta_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix}(\tau)\right)^* = \Delta_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix}(\tau)\,. \tag{4.7}$$

Using (4.7) and the definition of the projector (4.3) we obtain

$$\begin{aligned}\Delta_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix}(\tau) = {} & (3\delta_3(p)-1)\tilde{\Delta}\left(\frac{1}{3}|\tau\right) + (3\delta_3(p')-1)\tilde{\Delta}\left(\frac{\tau}{3}|\tau\right) + (3\delta_3(p+p')-1)\,\times \\ & \times\tilde{\Delta}\left(\frac{\tau-1}{3}|\tau\right) + (3\delta_3(p-p')-1)\tilde{\Delta}\left(\frac{\tau+1}{3}|\tau\right) - \ln(2\tau_2) + \tilde{c}\,. \end{aligned} \tag{4.8}$$

The term $\ln(2\tau_2)$ drops out of this expression all together, using the expression for the untwisted propagator (A.19).

The full string amplitude is defined by an integral over the fundamental domain. As the fundamental domain is symmetric under $\tau_1 \to -\tau_1$, it is important to know how the twisted propagators transform under this reflection. A straightforward analysis gives

$$\Delta_X \begin{bmatrix} p/3 \\ p'/3 \end{bmatrix}(-\tau_1, \tau_2) = \Delta_X \begin{bmatrix} \text{-}p/3 \\ p'/3 \end{bmatrix}(\tau_1, \tau_2) = \Delta_X \begin{bmatrix} p/3 \\ \text{-}p'/3 \end{bmatrix}(\tau_1, \tau_2)\,. \tag{4.9}$$

This shows that the twisted correlators with $p = 0$ or $q = 0$ are even under $\tau_1 \to -\tau_1$. From this discussion we conclude that there are four different propagators:

$$\begin{aligned} \Delta_u(\tau) &= \Delta_X \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} 0 \\ \text{-}1/3 \end{bmatrix}(\tau)\,, & \Delta_t(\tau) &= \Delta_X \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} \text{-}1/3 \\ 0 \end{bmatrix}(\tau)\,, \\ \Delta_{d_+}(\tau) &= \Delta_X \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} \text{-}1/3 \\ \text{-}1/3 \end{bmatrix}(\tau)\,, & \Delta_{d_-}(\tau) &= \Delta_X \begin{bmatrix} 1/3 \\ \text{-}1/3 \end{bmatrix}(\tau) = \Delta_X \begin{bmatrix} \text{-}1/3 \\ 1/3 \end{bmatrix}(\tau)\,, \end{aligned} \tag{4.10}$$

which we will use to characterize the sectors they come from: untwisted (u), twisted (t) and double twisted ($d_\pm$). It will be convenient to sometimes interpret these symbols u, t and $d_\pm$ as set of point (p, p'): $u = \{(0, \frac{1}{3}), (0, \frac{2}{3})\}$, $t = \{(\frac{1}{3}, 0), (\frac{2}{3}, 0)\}$, $d_+ = \{(\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})\}$, and $d_- = \{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$. The double twisted states have non-trivial periodicity conditions around both cycles of the world sheet torus. Using the classification we can write the expression for the tadpole profile as

$$G_j^{bJ}(k_6) = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \sum_{s=u,t,d_\pm} Q_s^{bJ}\, e^{-\Delta_s(\tau)k_i k_{\underline{i}}}\,, \qquad Q_s^{bJ} = \sum_{(p,p')\in s} \sum_{t_a, t'_a} \frac{1}{2}\frac{\partial}{\partial \nu_b^J} Z_{t_a, t'_a}^{p,p'}(\nu)\bigg|_0\,. \tag{4.11}$$

Because the fundamental domain is invariant under $\tau_1 \to -\tau_1$, the contributions of the sectors d_+ and d_- to the full integral are the same, since from (4.9) it follows that $\Delta_{d_+}(-\tau_1, \tau_2) = \Delta_{d_-}(\tau_1, \tau_2)$.

Another important thing we learn from this expression is that one has to require that all propagators $\Delta_s(\tau) > 0$ for all $\tau \in \mathcal{F}$: If there were to be a region of the fundamental domain $\mathcal{F}$ in which a propagator would be negative, it implies that the momentum profile function grows as a positive power of $\exp(k_i k_{\underline{i}})$, which is physically unacceptable because the Fourier transform to coordinate space does not exist. It can be shown that $\Delta_{d\pm}(\tau)$ takes the smallest value of all propagators at the two end points of the fundamental domain $\tau_\pm = (\mp 1 + \sqrt{3}\,i)/2$. This condition leads to a lower bound on the normal ordering constant:

$$\tilde{c} \geq \tilde{c}_0 = \ln \left| \frac{\vartheta_1^2(\frac{\tau_+ - 1}{3}|\tau_+)\vartheta_1'(0|\tau_+)}{\vartheta_1(\frac{\tau_+ + 1}{3}|\tau_+)\vartheta_1(\frac{\tau_+}{3}|\tau_+)\vartheta_1(\frac{1}{3}|\tau_+)} \right|^2 . \tag{4.12}$$

in terms of τ_- a similar expression can be given. We will argue in our next paper [20], which focusses more on the phenomenological aspects of the tadpoles, that the saturation of the bound might be preferred.

5. Model specific analysis

We also note that this normal ordering constant is a consequence of the (naive) procedure that we have used to extend the string amplitudes off-shell. To find a consistent definition of off-shell vertex operators is a complicated issue which requires further study, and is therefore beyond the scope of the present paper.

Our analysis has been essentially model independent up to this point. However, as has been investigated at length using field theory methods [17, 51] the local tadpoles associated with (anomalous) U(1)'s depend very sensitively upon the particular model examined. In table 1 we have summarized the five possible $\mathbb{C}^3/\mathbb{Z}_3$ models within the heterotic $\mathrm{E}_8 \times \mathrm{E}_8'$ string theory by giving their defining gauge shifts v_a and the unbroken gauge group in the effective four dimensional field theory of string zero modes. These gauge shifts are uniquely defined up to $\mathrm{E}_8 \times \mathrm{E}_8'$ root lattice shifts and Weyl reflections, which lead to complex conjugation of states in the string spectrum. From the field theory analysis we know that the only possible anomalous U(1) generators are $q_b = \sum_J v_b^J H_b^J$ where v_b^J is the gauge shift and H_b^J is an element of the Cartan subalgebra of the gauge group. In table 1 have given the traces of q_b of the untwisted and twisted zero modes. In string theory we probe the trace of these charges by calculating the expectation value of $q_b = \sum_J v_b^J V_j^{bJ}$.

All the classifying gauge shifts of table 1 can be represented as

$$v = \frac{1}{3}\left(1^{2r_1^1}\ \text{-}2^{r_1^2}\ 0^{8-2r_1^1-r_1^2}\ \middle|\ 1^{2r_2^1}\ \text{-}2^{r_2^2}\ 0^{8-2r_2^1-r_2^2}\right), \tag{5.1}$$

for some integers r_b^a. This shift fulfills the constraints (2.8) when $r_1^1 + r_2^1 = r_1^2 + r_2^2$. We apply a similar short-hand notation for products of r_b^a theta functions, and write

$$\vartheta\left[\begin{smallmatrix}\alpha\\ \alpha'\end{smallmatrix}\right](\nu^{r_b^a}|\tau) = \prod_{I\in r_b^a} \vartheta\left[\begin{smallmatrix}\alpha\\ \alpha'\end{smallmatrix}\right](\nu_b^I|\tau)\,, \tag{5.2}$$

identifying the index r_b^a with the corresponding set of 1's and -2's. Similarly, $8 - 2r_b^1 - r_b^2$ denotes the set of 0's. With this notation and the periodicities of the characteristics of the

model	gauge shift	gauge group	U(1) trace	
	$(v_1^I \mid v_2^I)$	G	$(q_1,q_2)_{un}$	$(q_1,q_2)_{tw}$
E_8	$\frac{1}{3}(\ 0^8 \mid 0^8\ )$	$\mathrm{E}_8 \times \mathrm{E}_8{}'$		
E_6	$\frac{1}{3}(\text{-}2,\ 1^2,\ 0^5 \mid 0^8\ )$	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8{}'$		
$\mathrm{E}_6{}^2$	$\frac{1}{3}(\text{-}2,\ 1^2,\ 0^5 \mid \text{-}2,\ 1^2,\ 0^5)$	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_6{}' \times \mathrm{SU}(3)'$		
E_7	$\frac{1}{3}(\ 0,\ 1^2,\ 0^5 \mid \text{-}2,\ 0^7\ )$	$\mathrm{E}_7 \times \mathrm{U}(1) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(6,2)$	$(10,\text{-}2)$
SU(9)	$\frac{1}{3}(\text{-}2,\ 1^4\ ,0^3 \mid \text{-}2,\ 0^7\ )$	$\mathrm{SU}(9) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	$(0,2)$	$(0,6)$

Table 1: The defining gauge shifs $(v_1|v_2)$ and the resulting unbroken zero mode gauge groups are displayed for the five $\mathbb{Z}_3$ orbifold models. The last two columns give the zero mode traces of the generators q_b over the untwisted (un) and twisted (tw) sectors, when applicable.

theta functions (B.3), the holomorphic partition function (2.11) reads

$$Z^{p,p'}_{t_a,t'_a}(\nu|\tau) = \frac{-1}{2\pi}\hat{\eta}^{p,p'}_{t_a,t'_a}\, \frac{\prod_a \vartheta\begin{bmatrix}\frac{1-t_a}{2}-\frac{a}{3}\\ \frac{1-t'_a}{2}-\frac{p'}{3}\end{bmatrix}(\nu^{2r^1_a+r^2_a}|\tau)\vartheta\begin{bmatrix}\frac{1-t_a}{2}\\ \frac{1-t'_a}{2}\end{bmatrix}(\nu^{8-2r^1_a-r^2_a}|\tau)}{\eta(\tau)^{15}\vartheta\begin{bmatrix}\frac{1}{2}-\frac{p}{3}\\ \frac{1}{2}-\frac{p'}{3}\end{bmatrix}^3(0|\tau)}\,. \qquad (5.3)$$

with the modified phase factor $\hat{\eta}^{p,p'}_{t_a,t'_a} = \tilde{\eta}^{p,p'}_{t_a,t'_a}\ \exp\bigl(2\pi i p'\bigl[\sum_a r^2_a(\frac{1-t_a}{2}-\frac{p}{3})+\frac{p}{3}-\frac{1}{2}\bigr]\bigr)$. The expectation value of this charge q_b can be conveniently computed in any particular sector as the derivative of the character-valued holomorphic partition function (2.11). This takes the form of the twisted fermionic correlator (A.10)

$$\begin{aligned} Q_b|^{p,p'}_{t_a,t'_a}(\tau) Z^{p,p'}_{t_b,t'_b}(\tau) &= \sum_J v^J_b \frac{\partial}{\partial\nu^J_b} Z^{p,p'}_{t_a,t'_a}(\nu|\tau)\Big|_0 \\ &= 2(r^1_b - r^2_b)\frac{\partial}{\partial\nu}\ln\vartheta\begin{bmatrix}\frac{1-t_a}{2}-\frac{p}{3}\\ \frac{1-t'_a}{2}-\frac{p'}{3}\end{bmatrix}(\nu|\tau)\Big|_0 Z^{p,p'}_{t_a,t'_a}(0|\tau)\,, \end{aligned} \qquad (5.4)$$

using the holomorphic partition function in the form (5.3).

Non-anomalous models. From the expression (5.4) we immediately conclude that the E_8, E_6 and $\mathrm{E}_6{}^2$ models, defined by the gauge shifts given in table 1, do not have any local (and therefore integrated) tadpoles in string theory, because they have the special property that $r^1_a = r^2_a$ for both $a = 1$ and 2. This is quite a remarkable result since this is not only a statement concerning the zero modes (from a four or ten dimensional point of view) of the string theory, but is exact and based entirely on the gauge shifts, not on any particular property of the amplitudes themselves. This gives a direct string confirmation of the field theory results presented in ref. [17] based on the zero mode spectrum only. Of course, one could argue that this was to be expected since the four dimensional gauge groups do not contain any U(1) factors, and certainly not any anomalous U(1). By contrast, it should be noted that, even though SU(9) is also a non-abelian group, we cannot use the same string argument to show that the trace of the U(1) generator for this group vanishes, since table 1 shows that $r^1_a \neq r^2_a$.

model	charge	u-sector	t-sector	d_+-sector	d_--sector
SU(9)	$Q_s^1(\tau)$	0	0	0	0
	$Q_s^2(\tau)$	2	2	2	2
E_7	$Q_s^1(\tau)$	$3+\ldots$	$\frac{1}{9}q^{-\frac{1}{3}}+\frac{5}{3}+\ldots$	$\frac{1}{9}e^{i\frac{4\pi}{3}}q^{-\frac{1}{3}}+\frac{5}{3}+\ldots$	$\frac{1}{9}e^{i\frac{2\pi}{3}}q^{-\frac{1}{3}}+\frac{5}{3}+\ldots$
	$Q_s^2(\tau)$	$2+\ldots$	$\frac{2}{9}q^{-\frac{1}{3}}-\frac{2}{3}+\ldots$	$\frac{2}{9}e^{i\frac{4\pi}{3}}q^{-\frac{1}{3}}-\frac{2}{3}+\ldots$	$\frac{2}{9}e^{i\frac{2\pi}{3}}q^{-\frac{1}{3}}-\frac{2}{3}+\ldots$

Table 2: The charges $Q_s^b(\tau)$ as functions of $q = \exp(2\pi i\,\tau)$ for the two anomalous models are displayed for the four different sectors u, t, d_+ and d_- defined in section 4. The results for the SU(9) model are exact, while for the E_7 model the dots indicates that all massive string states are neglected here. The tachyonic contribution within the various twisted sectors $(t, d_\pm)$ cancel among themselves.

Anomalous models. Next we move to the models that have an anomalous U(1) in their zero mode matter spectrum. To determine which states contribute to the local tadpoles at one loop, it is convenient to derive power series expansions of the functions $Q_s(\tau)$ in $q = \exp(2\pi i\,\tau)$. The masses of the relevant states are encoded as the power of q in these expansions. In particular, a negative power signals that tachyons give non-vanishing effects. In table 2 we only quote the leading order results; they already give an interesting insight in the contributing states, as we now discuss in detail for both anomalous models individually.

We begin with the situation in the SU(9) model. The absence of the dots is meant to indicate that results for the expressions for $Q_s(\tau)$ for the various sectors s are exact. Since there are only constants present, only massless string modes contribute to the gauge field tadpole. This suggests for this model the effective field theory approach, describing only the zero modes accounts for all contributions. Moreover, since the zero mode gauge group is $\mathrm{SU}(9) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$ and only zero modes contribute, it comes as no surprise that the traces of Q_s^1 vanish for all sector s separately. The traces of Q_s^2 are equal for all four sectors. Moreover, the sum of charges of the untwisted sector $(un = u)$ and the twisted sectors $(tw = t, d_+$ and $d_-)$ are equal to 2 and 6, respectively (see table 2), which agrees with results obtained in ref. [17, 51] quoted in the last two columns of table 1.

These results of the SU(9) model are in sharp contrast to the situation in the E_7 model as the bottom part of table 2 demonstrates. Their only similarity is that the E_7 model also passes the cross check that the sums of the zero mode charges of tables 1 and 2 agree. But in addition to the zero modes, whole towers of massive string states contribute to the gauge field tadpoles as the dots indicate. However, it can be shown that for the integrated tadpole, which simply sums up the contributions of all sectors, these massive excitations cancel out. More surprisingly, there are also tachyonic contributions to the different twisted sectors. It is not difficult to see from the table that also they cancel among themselves when one considers the integrated gauge field tadpole.

The situation in the E_7 model may be summarized as follows: At the four dimensional zero mode level, only the first U(1), generated by q_1, is anomalous. But locally we see that, because the various sectors have different profiles over the internal dimensions, both U(1) are anomalous and both tachyonic and massive states contribute to them. The conventional effective field theory description of this model is able to make the distinction between the momentum profiles due to the massless untwisted and the twisted states. Field theory does not determines its own spectrum, therefore it lacks the ability to predict the extra contributions of tachyonic (and massive string) states. In our next paper we will investigate how important of these massive and tachyonic states are for the final profile of the tadpoles.

In closing this section, we would like to remark that we have also computed the local tadpole in the SO(32) string. In this case the standard embedding ($v = (1^2,\text{-}2,0^{13}/3)$) gives rise to an anomalous U(1) generated by $q = w_I H_I^1$ with $w = (1^3,0^{13})/3$. The properties of the local gauge field tadpole are very similar to the SU(9) model of the $\mathrm{E}_8 \times \mathrm{E}_8{}'$ string: Also for the SO(32) string we found that only the zero modes have non-vanishing profiles over the extra dimensions. The integrated tadpole gives results consistent with the results quoted in ref. [7]

6. Conclusions

We have computed local gauge field tadpoles in heterotic $\mathrm{E}_8 \times \mathrm{E}_8{}'$ strings on the non-compact orbifold $\mathbb{C}^3/\mathbb{Z}_3$. This calculation confirms recent field theoretical calculation of such tadpoles, but at the same time extends these results in various interesting and surprising directions. In detail, our findings are the following:

The shape of these tadpoles are governed by the propagators of the twisted coordinate fields on the string world sheet: Their expectation values $\langle X^{\underline{i}} X^i \rangle(\tau)$ determine the widths of gaussian momentum distributions on the orbifold. The propagators can be classified according to their orbifold boundary conditions; the corresponding sectors give rise to gaussian distributions of various widths. The momentum distribution of the total tadpole is obtained by integrating these gaussians over the fundamental domain of the Teichmüller parameter of the one loop world sheet torus. This means that the tadpole in coordinate space is not a simple gaussian distribution, but a sum of gaussians weighted by the respective partition functions and this sum is integrated over the fundamental domain.

The propagators for the twisted coordinate fields X^i and $X^{\underline{i}}$ at zero separation are determined up to a single universal normal ordering constant associated with the subtraction of the logarithmic singularity of the untwisted correlators at zero separation. Conventional wisdom suggests that this normal ordering constant should be irrelevant as it drops out of on-shell string amplitudes. However, since the gauge field tadpole is necessarily an off-shell quantity, the final expression does depend on this normal ordering constant. Moreover, if this constant becomes too small, the coordinate space expression for the tadpole becomes ill-defined. This determines a lower bound for the value of this normal ordering constant.

We found that all eight bosonic propagators for the different twist sectors on the torus world sheet can be expressed in terms of four fundamental correlators that have different dependence on the modular parameter τ. This implies that both four dimensional untwisted and twisted sectors have distinct non-trivial profiles over the extra dimensions. In the field theory discussion of string orbifold models one usually assumes that the twisted sector states live exactly at the orbifold fixed points. Our calculation suggests that this is an approximation insensitive to string physics in which the twisted states are spread out around the orbifold singularity. One of the objectives of our follow-up paper [20] is to investigate just how crude the conventional field theory approximation really is; this leads some suggestions how fixed point states could be treated in field theoretical models.

Finally, we have investigated which states contribute in the different sectors to the momentum or coordinate profiles of the gauge field tadpoles. The anomalous $\mathbb{Z}_3$ model containing the gauge group SU(9) complies with the expectation that only the zero modes participate. However, quite surprisingly, we found that for the other anomalous model massive and even tachyonic string states contribute. We emphasize that this is not in contradiction with previous results in the literature for the zero mode D-term: We have verified that on-shell, which for a tadpole means that $k_i = 0$, the contributions from all tachyonic and massive states vanish. However, for generic $k_i \neq 0$ the effects of these states do not entirely cancel out when the four orbifold sectors are combined. In our next paper we explore the significance of these contributions further numerically.

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A. World sheet torus theories

In this appendix we collect some results concerning fermionic and bosonic conformal field theories on the torus world sheet. Most results are well-known, but for completeness and to fix our notations and conventions we review them here. More pedagogical discussions can be found in ref. [52]–[57]. Many of the properties of conformal field theories can be conveniently encoded by theta and related functions. We have summarized their properties in our conventions in appendix B.

The complex world sheet world sheet torus coordinate σ satisfies the periodicities $\sigma \sim \sigma + 1$ and $\sigma \sim \sigma + \tau$, where $\tau = \tau_1 + i\tau_2$ is the modular parameter labeling conformally distinct tori. The volume of the torus is $\int \mathrm{d}^2\sigma = 2\,\tau_2$. We define the functions

$$\Phi \left[{\alpha \atop \beta}\right] (\sigma, \bar\sigma | \tau) = \exp\left(2\pi i \, \frac{(\tau\alpha + \beta)\bar\sigma - (\bar\tau\alpha + \beta)\sigma}{\bar\tau - \tau} \right), \tag{A.1}$$

that have the following periodicity properties on the world sheet torus

$$\Phi \left[{\alpha \atop \beta}\right] (\sigma + 1) = e^{-2\pi i\,\alpha}\, \Phi \left[{\alpha \atop \beta}\right] (\sigma)\,, \qquad \Phi \left[{\alpha \atop \beta}\right] (\sigma + \tau) = e^{+2\pi i\,\beta}\, \Phi \left[{\alpha \atop \beta}\right] (\sigma)\,, \tag{A.2}$$

Here we have suppressed part of the arguments of these functions for brevity. Under modular transformations we have the transformations

$$\Phi \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\sigma|\tau+1) = \Phi \left[\begin{smallmatrix}\alpha\\ \beta+\alpha\end{smallmatrix}\right] (\sigma|\tau)\,, \qquad \Phi \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \left(\frac{\sigma}{\tau}\middle|\frac{-1}{\tau}\right) = \Phi \left[\begin{smallmatrix}+\beta\\ -\alpha\end{smallmatrix}\right] (\sigma|\tau)\,. \tag{A.3}$$

A.1 Complex fermion

Let λ be a complex fermion on the world sheet torus which obeys the boundary conditions

$$\lambda(\sigma+1) = e^{-2\pi i\,\alpha}\,\lambda(\sigma)\,, \qquad \lambda(\sigma+\tau) = e^{+2\pi i\,\beta}\,\lambda(\sigma)\,, \tag{A.4}$$

and the complex conjugate boundary conditions for $\bar\lambda$. The mode expansion for λ can be expressed using (A.1) and (A.2) as

$$\lambda(\sigma) = \sum_{m,n} \Phi \left[\begin{smallmatrix}m+\alpha\\ n\;+\beta\end{smallmatrix}\right] (\sigma)\, c^m_n\,, \tag{A.5}$$

where the sum is over all integers $m, n \in \mathbb{Z}$. The (quantized) harmonic oscillators are denoted by c^m_n. One can define free left-moving (holomorphic) theory for these boundaries by the actions

$$S_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\mu|\tau) = \frac{-1}{2\pi} \int \mathrm{d}^2\sigma \left(\bar\lambda\,\bar\partial\,\lambda + \frac{2\pi i}{\bar\tau-\tau}\mu\,\bar\lambda\lambda\right) = \sum_{m,n} \big[\tau(m+\alpha)+n+\beta+\mu\big]\,\bar c^{-m}_{-n} c^m_n\,. \tag{A.6}$$

Here we have introduced a source term μ, for later convenience. We note that because the spectrum of the right-moving (anti-holomorphic) theories (with S_ψ) is conjugate to the spectrum of the left-moving theory, we need only give the expressions for the holomorphic theory (S_λ) here. The (character valued) partition function [58] is given in terms of the theta function (B.1) and the Dedekind η-function (B.6) by

$$Z_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\mu|\tau) = e^{2\pi i\,\alpha\left(\frac{\tau}{2}\alpha+\beta+\mu-\frac{1}{2}\right)} \int \prod_{m,n} \mathrm{d}c^m_n \mathrm{d}\bar c^{-m}_{-n}\, e^{-S_\lambda\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\mu|\tau)} = \frac{\vartheta \left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right] (\mu|\tau)}{\eta(\tau)}\,. \tag{A.7}$$

This result is obtained by discarding (infinite) constant factors, ζ-function regularization techniques [54] and the product expansion of the theta function $\vartheta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ given in (B.2). The phase factor in front of the path integral has been chosen such that the final result can be written in terms of the theta function with characteristics $\frac{1}{2}-\alpha$ and $\frac{1}{2}-\beta$.

Using standard field theory techniques the propagator (for $\mu = 0$) can be determined

$$\tilde\Delta_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\sigma|\tau) = -\sum_{m,n} \frac{\Phi \left[\begin{smallmatrix}m+\alpha\\ n\;+\beta\end{smallmatrix}\right] (-\sigma|\tau)}{\tau(m+\alpha)+n+\beta} = -\frac{\vartheta_1'(0|\tau)\,\vartheta \left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right] (\sigma|\tau)}{\vartheta_1(\sigma|\tau)\,\vartheta \left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right] (0|\tau)}\,. \tag{A.8}$$

As usual the propagator only depends on the relative world-sheet separation. We use the notation $\tilde\Delta$ to denote two-point correlation functions at non-vanishing separation σ. The conformally normal ordered expression of the propagator is defined by [55, 56]

$$\Delta_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\tau) = \lim_{\sigma\to 0} \langle\bar\lambda(\sigma)\lambda(0)\rangle \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\tau) + \frac{1}{\sigma}\,, \tag{A.9}$$

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in the limit of zero separation $\sigma \to 0$. Using the theta function expression for the propagator (A.8), this becomes

$$\Delta_\lambda \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau) = -\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right]'(0|\tau)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right](0|\tau)} = -\frac{\partial}{\partial\mu}\ln Z_\lambda\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\mu|\tau)\bigg|_{\mu=0}. \tag{A.10}$$

The second expression gives an alternative way to derive this expectation value, using the character valued partition function (A.7).

A.2 Complex boson

We consider a complex boson X on the string world sheet with boundary conditions

$$X(\sigma+1) = e^{-2\pi i\,\alpha}X(\sigma)\,, \qquad X(\sigma+\tau) = e^{+2\pi i\,\beta}X(\sigma)\,. \tag{A.11}$$

As for the fermions, the mode expansion is

$$X(\sigma) = \sum_{m,n}\Phi\left[\begin{smallmatrix}m+\alpha\\ n\ +\beta\end{smallmatrix}\right](\sigma)\,a_n^m\,, \tag{A.12}$$

and the dynamics of the boson are described by the free action

$$S_X\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau) = \frac{-1}{2\pi}\int \mathrm{d}^2\sigma\Big(\partial\bar X\,\bar\partial X + \bar\partial\bar X\,\partial X\Big) = 2\pi\sum_{m,n}\frac{|\tau(m+\alpha)+n+\beta|^2}{2\tau_2}\,\bar a_{-n}^{-m}a_n^m\,. \tag{A.13}$$

The resulting partition function takes the form

$$Z_X\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau) = \left|e^{2\pi i\left\{\frac{1}{2}\alpha^2\tau+\alpha(\beta-\frac{1}{2})\right\}}\right|^2\int\prod_{m,n}\mathrm{d}a_n^m\mathrm{d}\bar a_{-n}^{-m}\,e^{-S_X\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau)} = \frac{\left|\eta(\tau)\right|^2}{\left|\vartheta\left[\begin{smallmatrix}\frac{1}{2}-\alpha\\ \frac{1}{2}-\beta\end{smallmatrix}\right](0|\tau)\right|^2}\,. \tag{A.14}$$

Here we have used the same phase factor as in (A.7). As for the fermions, the formal expression for the propagator reads

$$\tilde\Delta_X\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\sigma|\tau) = -\frac{1}{2\pi}\sum_{m,n}\frac{2\tau_2}{|\tau(m+\alpha)+n+\beta|^2}\Phi\left[\begin{smallmatrix}m+\alpha\\ n\ +\beta\end{smallmatrix}\right](-\sigma|\rho)\,. \tag{A.15}$$

For any of the boundary conditions (which we do not write explicitly here) one can derive that

$$\int\mathcal{D}X\,e^{ik\,\bar X(\sigma)+il\,\bar\partial\bar X(\sigma)+i\bar k\,X(\sigma')+i\bar l\,\bar\partial X(\sigma')} = e^{-\bar k k\,\tilde\Delta_X+k\bar l\,\bar\partial\tilde\Delta_X-\bar k l\,\bar\partial\tilde\Delta_X+\bar l l\,\bar\partial\bar\partial\tilde\Delta_X}(\sigma'-\sigma) \tag{A.16}$$

for arbitrary $k, \bar k, l$ and $\bar l$.

The properties of the twisted propagators are of central importance to our work, and are, therefore, discussed in section 4 of our main discussion. As these propagators can be expressed in terms of the untwisted bosonic propagator, with $\alpha = \beta = 0$, we review its properties in this appendix. The formal series expansion of the untwisted propagator reads

$$\tilde{\Delta}(\sigma|\tau) = -\frac{1}{2\pi} \sum_{m,n}{}' \frac{2\tau_2}{|\tau m + n|^2} \, \Phi \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] (-\sigma|\tau) \,, \tag{A.17}$$

where the prime on the sum indicates that the sum is over all integers with $(m,n) \neq (0,0)$. It follows that the regularized correlator is a solution of

$$\bar{\partial}\partial \, \tilde{\Delta}(\sigma|\tau) = 2\pi \left(\delta^2(\sigma) + \frac{1}{2\tau_2} \right) , \tag{A.18}$$

which is required to be modular invariant and periodic. Here we have chosen the same normalization for the bosonic propagator as for the fermionic propagator with respect to the delta function $\delta^2(\sigma)$ in the defining differential equation (A.9). This correlator can be expressed in terms of theta functions as

$$\tilde{\Delta}(\sigma|\tau) = -\ln \tilde{G}(\sigma|\tau) \,, \qquad \tilde{G}(\sigma|\tau) = 2\tau_2 \, e^{-2\pi \frac{\sigma_2^2}{\tau_2}} \left| \frac{\vartheta_1(\sigma|\tau)}{\vartheta_1'(0|\tau)} \right|^2 . \tag{A.19}$$

Notice that this fixes $\tilde{\Delta}$ up to an additive constant. (Fortunately, for the determination of the twisted propagator this undetermined constant is irrelevant, see section 4.) Finally, the normal ordered untwisted propagator at zero separation is given by

$$\Delta(\tau) = \lim_{\sigma \to 0} \langle : \bar{X}(\sigma) X(0) : \rangle \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] (\tau) = \tilde{\Delta}(\sigma|\tau) + \ln|\sigma|^2 + \tilde{c} = -\ln(2\tau_2) + \tilde{c} \,, \tag{A.20}$$

with $\tilde{c}$ a normal ordering constant.

B. Theta functions

The genus one theta function is defined by

$$\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] (\sigma|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\alpha)^2} \, e^{2\pi i \, (\sigma - \beta)(n-\alpha)} \,, \qquad q = e^{2\pi i \, \tau} \,. \tag{B.1}$$

In a product representation it takes the form:

$$\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] (\sigma|\tau) = e^{-2\pi i \alpha(\sigma - \beta)} \, q^{\frac{1}{2}\alpha^2} \prod_{n \geq 1} \left\{ \left(1 - q^n\right) \prod_{s = \pm} \left(1 + e^{2\pi i s(\sigma-\beta)} \, q^{n - \frac{1}{2} - s\alpha}\right) \right\} . \tag{B.2}$$

The arguments of the theta functions are periodic in the sense that

$$\vartheta \left[\begin{smallmatrix} \alpha+1 \\ \beta \end{smallmatrix}\right] (\sigma|\tau) = \vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] (\sigma|\tau) \,, \qquad \vartheta \left[\begin{smallmatrix} \alpha \\ \beta+1 \end{smallmatrix}\right] (\sigma|\tau) = e^{2\pi i \, \alpha} \, \vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] (\sigma|\tau) \,. \tag{B.3}$$

Modular transformations have the following effect on the theta functions

$$\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] (\sigma|\tau+1) = e^{-\pi i \, \alpha(\alpha+1)} \, \vartheta \left[\begin{smallmatrix} \alpha \\ \beta+\alpha+\frac{1}{2} \end{smallmatrix}\right] (\sigma|\tau) \,,$$

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$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{\sigma}{\tau} \Big| \frac{-1}{\tau} \right) = \sqrt{-i\tau}\, e^{2\pi i \left\{ \frac{\sigma^2}{2\tau} + \alpha\beta \right\}} \vartheta \begin{bmatrix} +\beta \\ -\alpha \end{bmatrix} (\sigma|\tau) \,. \tag{B.4}$$

The periodicities of the theta functions read

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma + 1|\tau) = e^{-2\pi i\, \alpha}\, \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau) \,, \qquad \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma + \tau|\tau) = e^{2\pi i\, (\beta - \sigma - \frac{1}{2}\tau)}\, \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\sigma|\tau) \,. \tag{B.5}$$

An often used notation is $\vartheta_1 = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, $\vartheta_2 = \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$, $\vartheta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\vartheta_4 = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. Another important modular function is the Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} \left(1 - q^n \right) \,, \qquad \vartheta_1'(0|\tau) = 2\pi \left(\eta(\tau) \right)^3 \,, \tag{B.6}$$

where the prime $'$ denotes differentiation by the first argument of ϑ_1. The modular transformation properties of the Dedekind function take the form

$$\eta(\tau + 1) = e^{2\pi i \frac{1}{24}} \eta(\tau) \,, \qquad \eta \left(\frac{-1}{\tau} \right) = \sqrt{-i\tau}\, \eta(\tau) \,. \tag{B.7}$$

The Riemann's identity reads

$$\sum_{s,s'=0,1} e^{-\pi i\, s s'} \prod_{i=0}^{3} \vartheta \begin{bmatrix} \frac{1}{2} - \alpha_i - \frac{s}{2} \\ \frac{1}{2} - \beta_i - \frac{s'}{2} \end{bmatrix} (\mu_i) = 2 \prod_{i=0}^{3} \vartheta \begin{bmatrix} \frac{1}{2} + \alpha_i \\ \frac{1}{2} + \beta_i \end{bmatrix} (\tilde{\mu}_i) = 2 \prod_{i=0}^{3} \vartheta \begin{bmatrix} \frac{1}{2} - \alpha_i \\ \frac{1}{2} - \beta_i \end{bmatrix} (-\tilde{\mu}_i) \,, \tag{B.8}$$

with $\tilde{\mu}_i = \frac{1}{2} \sum_k \mu_k - \mu_i$ and $\frac{1}{2} \sum_i \alpha_i = \frac{1}{2} \sum_i \beta_i = 0 \mod 1$.

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